A consequence of the zero-fourth-cumulant approximation in the decay of isotropic turbulence

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This paper is a continuation of previous work (Ogura 1962*a*, *b*) on the dynamical consequence of the hypothesis that fourth-order mean values of the fluctuating velocity components are related to second-order mean values as they would be for a normal joint-probability distribution. The equations derived by Tatsumi (1957) for isotropic turbulence on the basis of this hypothesis are integrated numerically for specific initial conditions. The initial values of the Reynolds number, $R_\lambda = (\overline{u^2})^{\frac{1}{2}}\,\lambda/\nu$, assigned in this investigation are 28·8, 14·4, 7·2 and 1·8, where $(\overline{u^2})^{\frac{1}{2}}$ is the root-mean-square turbulent velocity, λ the dissipation length and ν the kinematic viscosity coefficient.

The result of such computations is that the energy spectrum does develop negative values for $R_\lambda = 28{\cdot}8$ and 14·4. This first occurs at a time approximately 2·8 for $R_\lambda = 28{\cdot}8$ and 4·2 for $R_\lambda = 14{\cdot}4$. The time-scale here is $(E_0\kappa_0^3)^{-\frac{1}{2}}$, where κ_0 is a wave-number scale typical of the energy-containing velocity component and E_0, a typical value of the energy spectrum, is given by $4\pi^{-\frac{1}{2}}\kappa_0^{-1}\overline{u^2}$.

There is no evidence of the energy distribution tending to become negative for $R_\lambda = 7{\cdot}2$ and 1·8. It is observed that inertial effects are relatively weak at $R_\lambda = 7{\cdot}2$ and the decay process is largely controlled by viscous effects. For $R_\lambda = 1{\cdot}8$, a purely viscous calculation is found to be adequate to account for the numerically integrated results.

1. Introduction

The present paper is a continuation of two earlier papers by the writer on some consequences of the zero-fourth-cumulant approximation in the theory of energy transfer in incompressible, isotropic turbulence. In the first of these (Ogura 1962*a*), the dynamical equations which describe the behaviour of isotropic turbulence in two dimensions were derived on the basis of an hypothesis of Millionshtchikov (1941*a*, *b*). This hypothesis assumes that the relationship among mean values of quadruple velocity-component products and those of double velocity-component products is the one appropriate to a jointly normal probability distribution. The equations were then integrated numerically for specific initial conditions. The result revealed that the energy-spectrum function assumes negative values after a finite time. Truncation errors which arise from finite-difference approximations in numerical integration were examined and it was concluded that the negative values of the spectrum function cannot possibly be generated by truncation errors.

This study was extended in the second paper (Ogura 1962*b*, hereafter referred to as paper A), in which the equations derived independently by Proudman & Reid (1954) and by Tatsumi (1957) for three-dimensional turbulence were integrated numerically, but only for an inviscid fluid. While the generation of negative energy densities was again observed, it was not conclusively clear in this case whether or not this unphysical result was produced by the above hypothesis. This was because a larger error was induced in numerical calculations by truncating the infinite wave-number space at a finite wave-number.

The purpose of this paper is to present results of numerical integration of Tatsumi's equations in a viscous case for a particular set of initial conditions. The large truncation error mentioned above is no longer present. The computations are repeated for several different values of the Reynolds number, and negative energy densities are observed for all but the two lowest Reynolds numbers.

In connexion with the above hypothesis, it should be mentioned that Kraichnan (1962) has recently made a critical appraisal of the general cumulant-discard approximation, of which the above hypothesis is a particular case. He showed analytically that this hypothesis leads to a negative-definite power spectrum when it is applied to the 'convection' of a *scalar* field by a prescribed random velocity field. This phenomenon has been also observed by O'Brien & Francis (1962) in their work on the numerical integration of the appropriate spectral equations for a *scalar* variable.

2. The basic equations

The basic equations which we deal with in this paper are the same as those in paper A. For convenience of discussions which follow, some of them are summarized briefly in this section.

The dynamical equations derived by Tatsumi (1957) are written as

$$\frac{\partial E(\kappa,t)}{\partial t} + 2\nu\kappa^2 E(\kappa,t) = \int_0^\infty \int_{|\kappa-\kappa'|}^{\kappa+\kappa'} \Psi(\kappa,\kappa',\kappa'',t)\,d\kappa''\,d\kappa', \tag{2.1}$$

$$\begin{aligned} \partial\Psi/\partial t + \nu(\kappa^2+\kappa'^2+\kappa''^2)\Psi = {} & \phi_1(\kappa,\kappa',\kappa'')\,E(\kappa,t)\,E(\kappa',t) \\ & + \phi_2(\kappa,\kappa',\kappa'')\,E(\kappa',t)\,E(\kappa'',t) \\ & + \phi_3(\kappa,\kappa',\kappa'')\,E(\kappa'',t)\,E(\kappa,t), \end{aligned} \tag{2.2}$$

where $E(\kappa,t)$ is the energy spectrum function, ν the kinematic viscosity coefficient, κ the wave-number, t the time,

$$\phi_1 = (q/16\kappa^3\kappa'^3\kappa'')(-3\kappa^2\kappa''^2 + 2\kappa^2\kappa'^2 - \kappa'^4 - \kappa^4 + \kappa'^2\kappa''^2),$$

$$\phi_2 = (q/16\kappa\kappa'^3\kappa''^3)(3\kappa^2\kappa''^2 - 2\kappa'^2\kappa''^2 + \kappa''^4 + \kappa'^4 - \kappa'^2\kappa^2),$$

$$\phi_3 = (q/16\kappa^3\kappa'^3\kappa'')(\kappa^4 - \kappa^2\kappa'^2 - \kappa''^4 + \kappa'^2\kappa''^2),$$

and q is the symmetric quartic

$$q = 2\kappa^2\kappa'^2 + 2\kappa'^2\kappa''^2 + 2\kappa''^2\kappa^2 - \kappa^4 - \kappa'^4 - \kappa''^4.$$

The set (2.1) and (2.2) constitute the fundamental equations for the present study of turbulence. In numerical integrations the infinite integration in (2.1) is necessarily truncated at a finite limit, κ^* say.

For convenience of numerical analysis, dimensionless variables are introduced in the following form

$$\left.\begin{aligned} & t = \tau(\Delta t) \quad (\tau = 0, 1, 2, \ldots); \\ & \kappa = k(\Delta\kappa), \quad \kappa' = j(\Delta\kappa), \quad \kappa'' = i(\Delta\kappa), \quad \kappa^* = I(\Delta\kappa) \quad (i, j, k = 0, 1, 2, \ldots, I); \\ & E = E_0 \hat{E}(k, \tau), \quad \Upsilon = \Upsilon_0 \psi(k, j, i, \tau), \end{aligned}\right\} \tag{2.3}$$

where E_0 and Υ_0 are constants. Δt and $\Delta\kappa$ denote the finite-difference increments for time and wave-number, respectively. Equations (2.1) and (2.2) then take the following dimensionless finite-difference form

$$\begin{aligned} \hat{E}^{(\tau+1)}(k) = {} & \left[1 - \frac{k^2}{R_e}\right]\left[1 + \frac{k^2}{R_e}\right]^{-1} \hat{E}^{(\tau)}(k) \\ & + \sigma_1 \left[1 + \frac{k^2}{R_e}\right]^{-1} \int_0^I \int_{|k-j| \leqslant I}^{k+j \leqslant I} \psi^{(\tau+\frac{1}{2})}(k, j, i, \tau)\, di\, dj, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \psi^{(\tau+\frac{1}{2})} = {} & [1 - (1/2R_e)(k^2 + j^2 + i^2)][1 + (1/2R_e)(k^2 + j^2 + i^2)]^{-1} \psi^{(\tau-\frac{1}{2})} \\ & + \sigma_2 [1 + (1/2R_e)(k^2 + j^2 + i^2)]^{-1} [\hat{\phi}_1 \hat{E}^{(\tau)}(k) \hat{E}^{(\tau)}(j) \\ & + \hat{\phi}_2 \hat{E}^{(\tau)}(j) \hat{E}^{(\tau)}(i) + \hat{\phi}_3 \hat{E}^{(\tau)}(i) \hat{E}^{(\tau)}(k)], \end{aligned} \tag{2.5}$$

where a typical term like $\hat{E}^{(\tau)}(k)$ represents a value of $\hat{E}$ at $t = \tau(\Delta t)$ and $\kappa = k(\Delta\kappa)$, and the dimensionless forms of ϕ_1, ϕ_2 and ϕ_3 are obtained simply by replacing κ, κ' and κ'' by k, j and i, respectively. The dimensionless parameters in (2.4) and (2.5) are

$$R_e = \nu^{-1}(\Delta t)^{-1}(\Delta\kappa)^{-2}, \quad \sigma_1 = (\Delta t)(\Delta\kappa)^2 E_0^{-1} \Upsilon_0, \quad \sigma_2 = (\Delta t)(\Delta\kappa) E_0^2 \Upsilon_0^{-1}. \tag{2.6}$$

3. Results of numerical integrations

Equations (2.1) and (2.2) will yield information about the time and wave-number behaviour of an energy spectrum function $E(\kappa, t)$ only when two initial conditions $E(\kappa, 0)$ and $\Upsilon(\kappa, \kappa', \kappa'', 0)$ are specified. It would evidently be useful to examine a wide range of initial conditions to explore the degree of dependence of the solution on the initial statistical distribution. We will, however, consider only the simplest situation, that in which initially the statistical distribution of third-order moments are zero and hence $\Upsilon(\kappa, \kappa', \kappa'', 0)$ is zero. One spectrum for which $\Upsilon(\kappa, \kappa', \kappa'', 0)$ might be expected to be zero is the one typical of the final period of decay (Proudman & Reid 1954) when there is, in fact, negligible energy transfer. It should be emphasized that use of such a spectrum does not limit us to the final period, but merely provides us with a set of consistent and well-behaved initial conditions.

With these ideas in mind the spectrum whose decay we have computed has the following description

$$\hat{E}^{(0)}(k) = (k/k_0)^4 \exp[-(k/k_0)^2], \tag{3.1}$$

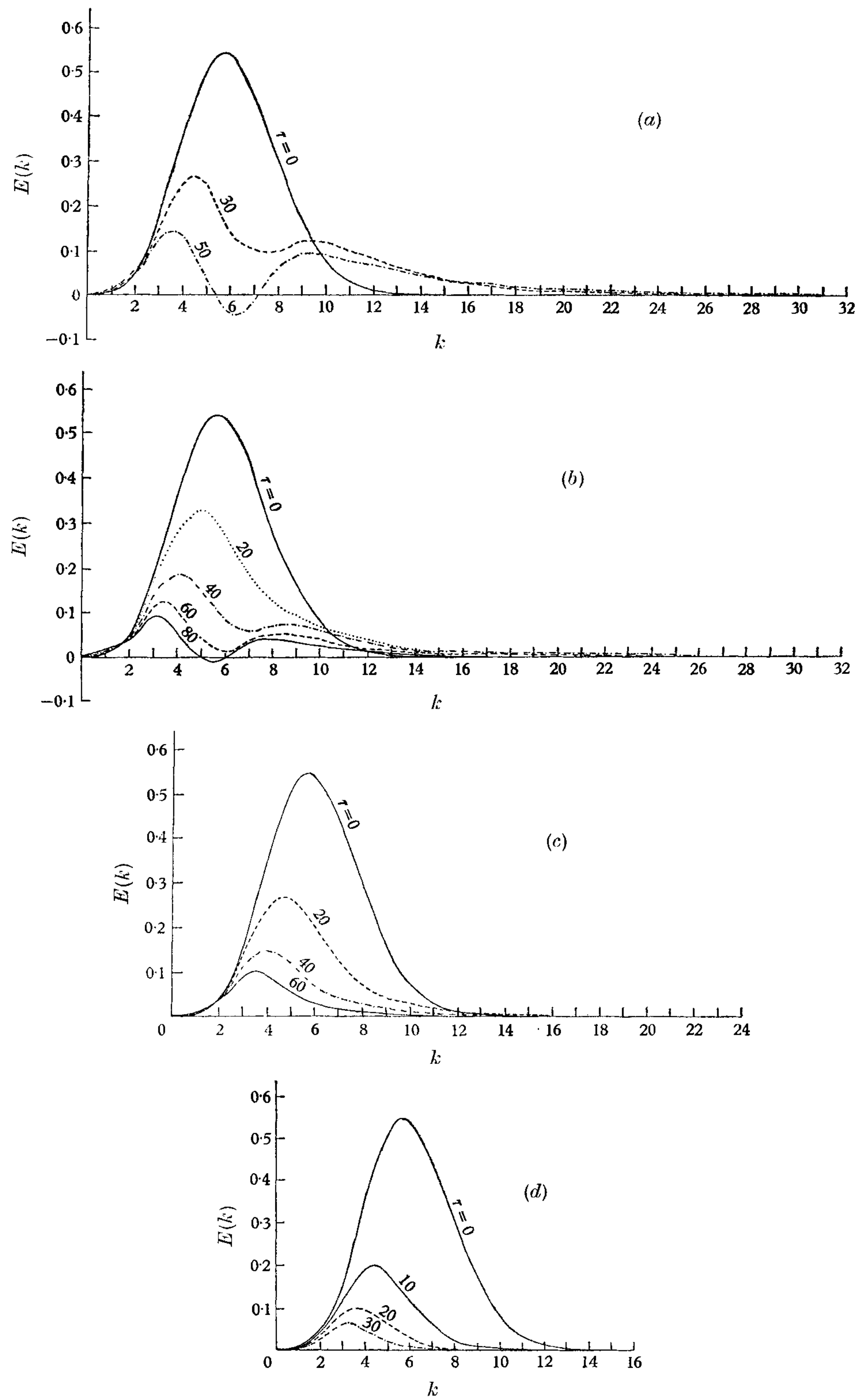

FIGURE 1. Variations of energy spectrum with time for various values of the Reynolds number. (a) $R_\lambda(0) = 28{\cdot}8$ ($R_e = 8000$); (b) $R_\lambda(0) = 14{\cdot}4$ ($R_e = 4000$); (c) $R_\lambda(0) = 7{\cdot}2$ ($R_e = 2000$); (d) $R_\lambda(0) = 1{\cdot}8$ ($R_e = 500$).

where $k_0 = \kappa_0/\Delta\kappa$ and κ_0 is a constant. Four different integrations of (2.1) and (2.2) have been completed, starting from the identical initial condition (3.1), for four different values of the Reynolds number, R_e; $R_e = 500$, 2000, 4000 and 8000. The following values are assigned to the dimensionless parameters

$$k_0 = 4, \quad I = 32, \quad \sigma_1 = 0{\cdot}0166, \quad \sigma_2 = 0{\cdot}003538. \tag{3.2}$$

The initial condition (3.1) is the same as that in paper A, apart from a constant factor. The results of numerical calculations are presented in figures 1(*a*)–(*d*) in which the dimensionless energy spectrum function (E) is plotted against the dimensionless wave-number (k) for various values of time.

In discussing these results, it is found to be convenient to introduce the following dimensionless parameters. First, instead of R_e in (2.6), we will use $R_\lambda(0)$ which represents the initial value of the Reynolds number in terms of the initial condition (3.1)

$$R_\lambda(0) = [(\overline{u^2})^{\frac{1}{2}}\lambda/\nu]_{t=0} = (\tfrac{1}{2}\sqrt{\pi})^{\frac{1}{2}}(E_0/\kappa_0)^{\frac{1}{2}}/\nu,$$

where λ denotes the dissipation length and $\overline{u^2}$ is the mean-square turbulent velocity. The equation above is derived from the following two equations

$$[\tfrac{3}{2}\overline{u^2}]_{t=0} = E_0\kappa_0 \int_0^\infty x^4 e^{-x^2}dx,$$

and

$$[15\nu(\overline{u^2}/\lambda^2)]_{t=0} = 2\nu E_0\kappa_0^3 \int_0^\infty x^6 e^{-x^2}dx.$$

The values of R_e assigned above are converted approximately to $R_\lambda(0) = 1{\cdot}8$, 7·2, 14·4 and 28·8, respectively.

Next, instead of τ in (2.3) we introduce a new time-scale $t_0 = (E_0\kappa_0^3)^{-\frac{1}{2}}$ and a new dimensionless time $t' = t/t_0$. From (3.2), the relation between t' and τ is derived as $t' = 0{\cdot}0614\tau$. It was noted in paper A that the mean-square vorticity would become infinite in an inviscid fluid after a time $\tau = 46$, which corresponds roughly to $t' = 2{\cdot}82$ on the new time-scale. It is also noted that this time may be compared with the time $t' = 0{\cdot}5$ required for the growth of the triple correlation from an initial value of zero (Proudman & Reid 1954).

Figures 2 and 3 show respectively the variations of the mean-square vorticity ($\overline{\omega^2}/\overline{\omega_0^2}$) and the mean-square velocity ($\overline{u^2}/\overline{u_0^2}$) as functions of time and the initial Reynolds number. Both quantities are represented as ratios with respect to their initial values. In cases where only the viscous terms are retained in the dynamical equations, the variations of $\overline{\omega^2}/\overline{\omega_0^2}$ and $\overline{u^2}/\overline{u_0^2}$ with time are given by

$$\frac{\overline{\omega^2}}{\overline{\omega_0^2}} = \left[1 + \frac{(2\sqrt{\pi})^{\frac{1}{2}}}{R_\lambda(0)}t'\right]^{-\frac{7}{2}},$$

and

$$\frac{\overline{u^2}}{\overline{u_0^2}} = \left[1 + \frac{(2\sqrt{\pi})^{\frac{1}{2}}}{R_\lambda(0)}t'\right]^{-\frac{5}{2}},$$

respectively. The relationships represented by these equations are also included in figures 2 and 3 for reference. For $R_\lambda(0) = 1{\cdot}8$, the viscous effects are found to predominate over the inertial effects so much that the two lines in these diagrams are almost indistinguishable.

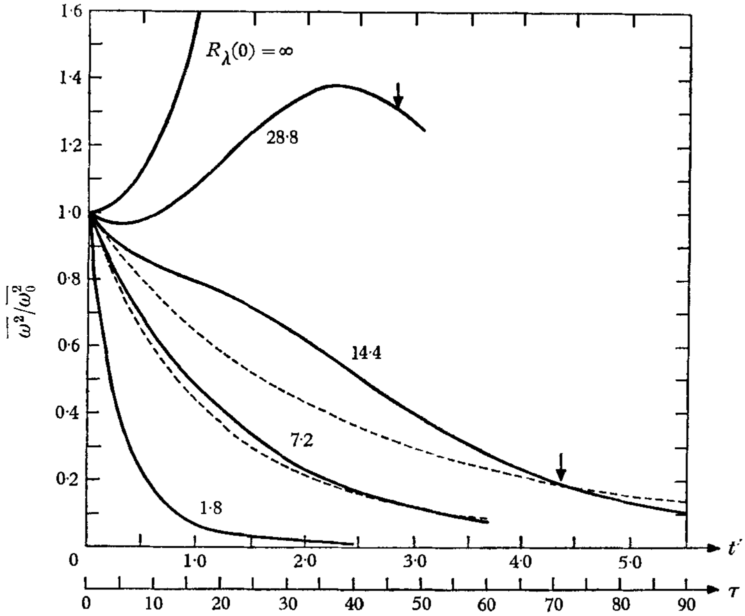

FIGURE 2. The mean-square vorticity as a function of time for various values of the Reynolds number (solid lines). Arrows mark the times at which the energy distributions take on negative values. The broken lines represent purely viscous calculations.

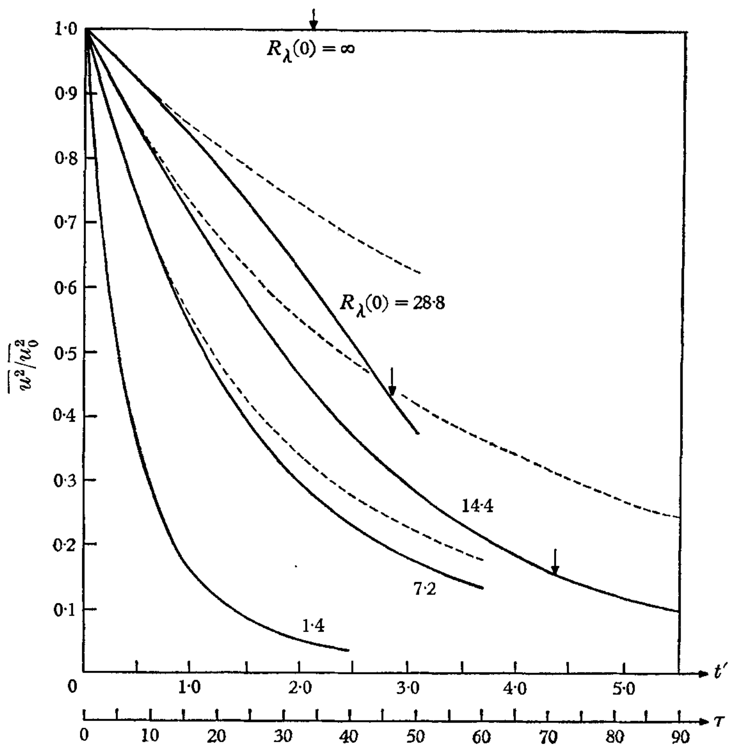

FIGURE 3. The same as figure 2, but for the mean-square turbulent velocity.

4. Discussion and conclusions

The most striking feature we observe in figure 1 is that, for $R_\lambda(0) = 28{\cdot}8$ and 14·4, the energy distribution develops negative values after a finite time (see table 1). For $R_\lambda(0) = \infty$, it was found difficult in paper A to control the accuracy of the numerical calculations due to the transfer of energy to large wave-numbers. As we observe in figures 1 (*a*) and (*b*), no appreciable amount of energy appears at the end of our finite wave-number domain in the present cases. This is due to the strong viscous effects operating at large wave-number, indicating that the numerical error induced by truncating the infinite wave-number space at a finite wave-number is absent in the present calculations.

$R_\lambda(0)$	∞	28·8	14·4
t'	2·08*	2·82	4·36

* From paper A.

TABLE 1. The time at which the energy density first becomes negative for various values of the Reynolds number

Figure 2 shows that, for $R_\lambda(0) = 28{\cdot}8$, the mean-square vorticity decreases slightly at first due to viscous effects. It then increases, as the transfer function develops with time, reaching a maximum value of $1{\cdot}38\,\overline{\omega_0^2}$ at $t' = 2{\cdot}3$. It thereafter decreases until the energy distribution takes on negative values at $t' = 2{\cdot}82$. For $R_\lambda(0) = 14{\cdot}4$, however, the mean-square vorticity is monotonically decreasing and negative values of the energy density do not appear until $t' = 4{\cdot}36$, at which time $\overline{\omega^2}$ has decreased to about $0{\cdot}2\,\overline{\omega_0^2}$. For $R_\lambda(0) = 7{\cdot}2$, there is no evidence of the energy distribution tending to become negative even after a time $t' = 3{\cdot}7$. However, at this low Reynolds number the decay of vorticity and energy are largely controlled by viscous effects. This is particularly true for $R_\lambda(0) = 1{\cdot}8$ where the numerically integrated results are found to agree almost completely with purely viscous calculations.

Thus, from the work mentioned in § 1 and also from the present investigation, it is now conclusively clear that the transfer theory based upon the zero-fourth-cumulant approximation does lead to negative energy densities at large Reynolds numbers.

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